4
Chapter

Archimedes' Determination of Circular Area
(ca. 225 B.C.)

The Life of Archimedes

Two to three generations separated Euclid from the next great mathematician on our agenda, the incomparable Archimedes of Syracuse (287–212 B.C.). By the end of his brilliant career, Archimedes had pushed mathematics well beyond the frontiers of Euclid's day. Indeed, the mathematical world would not see his like again for almost 2000 years.

We are fortunate to have a bit of information about Archimedes' life, although, as with any details coming to us over so many generations, its literal validity can often be challenged. A number of his mathematical works, often prefaced by his own commentaries, have also survived. Taken together, these resources give us a picture of a much revered, somewhat eccentric genius who dominated the mathematical landscape of the classical world.

Archimedes was born at Syracuse on the island of Sicily. His father is thought to have been an astronomer, and as a young boy, Archimedes developed a life-long interest in the study of the heavens. In his youth, Archimedes also spent some time in Egypt, where he appears to have studied at the great Library of Alexandria. This, of course, had been Euclid's base of operations, and Archimedes would naturally have been trained in the Euclidean tradition, a fact readily apparent in his own mathematical writings.

During his time in the Nile Valley, Archimedes is said to have invented the so-called "Archimedean screw," a device for raising water from a low level to a higher one. Interestingly, this invention remains in use to this day. Its creation testifies to the dual nature of Archimedes' genius: he could concern himself with practical, down-to-earth matters, or could delve into the most abstract, ethereal realms. In spite of Alexandria's obvious appeal to one of his scholarly talents, Archimedes chose to return to his native Syracuse and there, as far as can be determined, spent the rest of his days. Although isolated in Syracuse, he maintained a wide correspondence throughout the Greek world, and particularly with scholars at Alexandria. It is through such correspondence that much Archimedean material has survived.

His awesome mathematical talent was augmented by an ability to devote himself single-mindedly to any problem at hand in extraordinary periods of intense, focused concentration. At such times, the more mundane concerns of life were simply ignored. We learn from Plutarch that Archimedes would

> . . . forget his food and neglect his person, to that degree that when he was occasionally carried by absolute violence to bathe or have his body anointed, he used to trace geometrical figures in the ashes of the fire, and diagrams in the oil on his body, being in a state of entire preoccupation, and, in the truest sense, divine possession with his love and delight in science.

This passage portrays the stereotypically absent-minded mathematician, not to mention one to whom cleanliness was next to irrelevant. Of course, the most famous "absent-minded" story concerns the crown of King Hieron of Syracuse. The King, suspicious that his goldsmith had substituted some lesser alloy for the crown's gold, asked Archimedes to determine its true composition. As the story goes, Archimedes wrestled with the problem until one day (during what must have been one of his rare baths) he hit upon the solution. Jumping from the bath, he ran through the streets of Syracuse shouting "Eureka! Eureka!" Unfortunately, so absorbed was he in his wonderful discovery that he forgot to don his toga. What the townspeople thought at seeing their fellow citizen running stark naked in their midst is impossible to say.

This tale may be fictitious, but Archimedes' discovery of the fundamental principles of hydrostatics is pure fact. He left us a treatise titled

On Floating Bodies developing his ideas in this area. Additionally, he advanced the science of optics and did pioneering work in mechanics, as is evident not only in his water pump but in his wonderful understanding of the workings of levers, pulleys, and compound pulleys. Plutarch included the story of a skeptical King Hieron doubting the power of these simple mechanical devices. The King asked for a practical demonstration, and Archimedes obliged in dramatic fashion. He selected one of the King's largest ships

> . . . which could not be drawn out of the dock without great labour and many men; and, loading her with many passengers and a full freight, sitting himself the while far off, with no great endeavour, but only holding the head of the pulley in his hand and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea.

Needless to say, the King was impressed. Perhaps he sensed in this gifted scientist a valuable resource in the event that such engineering talents should be needed for more pressing matters. And indeed they were, when Rome, under the generalship of Marcellus, attacked Syracuse in 212 B.C. In the face of the Roman threat, Archimedes rose to the defense of his homeland by designing an array of weapons of great effectiveness. In the process, he became what can only be called a one-man military-industrial complex.

In what follows, we continue to quote liberally from Plutarch's *Life of Marcellus*, written by the great Roman biographer almost three centuries after the fact. While it was Marcellus about whom Plutarch was ostensibly writing, his admiration for Archimedes was quite evident. These writings provide us with an intriguing—and certainly a very colorful—account of Archimedes in action.

"Marcellus moved with his whole army to Syracuse," Plutarch wrote, "and encamping near the wall, sent ambassadors into the city." When the Syracusans refused to surrender, Marcellus opened his attack on the city walls, both on the land side with his troops and on the ocean side with 60 heavily armed galleys. Marcellus was counting on ". . . the abundance and magnificence of his preparations, and on his own previous glory," but he would prove no match for Archimedes and his diabolical war machines.

According to Plutarch, the Roman legions marched to the city walls, believing themselves to be invincible.

> But when Archimedes began to ply his engines, he at once shot against the land forces all sorts of missile weapons, and immense masses of stone that came down with incredible noise and violence; against which no man could

stand; for they knocked down those upon whom they fell in heaps, breaking all their ranks and files.

The Roman naval forces fared no better, for

> . . . huge poles thrust out from the walls over the ships sunk some by the great weights which they let down from on high upon them; others they lifted up into the air by an iron hand or beak . . . and, when they had drawn them up by the prow, and set them on end upon the poop, they plunged them to the bottom of the sea; or else the ships, drawn by engines within, and whirled about, were dashed against steep rocks that stood jutting out under the walls, with great destruction of the soldiers that were aboard them.

Such destruction, related Plutarch, was "a dreadful thing to behold," and one is inclined to agree. Under the circumstances, Marcellus thought it prudent to retreat. He withdrew both land and naval forces to regroup. Holding a council of war, the Romans decided upon a night assault, in the expectation that Archimedes' devilish weapons would be useless if the attackers slipped too close to the walls under the cover of darkness. Again, the Romans had an unpleasant surprise. The diligent Archimedes had arranged his devices for just such an eventuality, and no sooner had the Romans crept up close upon the fortifications than "stones came tumbling down perpendicularly upon their heads, and, as it were, the whole wall shot out arrows at them." In response, the terrified Romans again retreated, only to come under attack from Archimedes' longer-range weapons, an attack that "inflicted a great slaughter among them." By this time, the vaunted Roman legions, "seeing that indefinite mischief overwhelmed them from no visible means, began to think they were fighting with the gods."

It is perhaps an understatement to say that Marcellus had a serious morale problem. He demanded of his shaken troops a renewed courage to continue the assault, but the previously invincible Romans wanted no more of it. On the contrary, the soldiers "if they did but see a little rope or a piece of wood from the wall, instantly crying out, that there it was again, Archimedes was about to let fly some engine at them, they turned their backs and fled." Knowing that discretion is the better part of valor, Marcellus chose to abandon the direct assault.

Instead, trying to starve the trapped Syracusans into surrender, the Romans began a long siege of the city. Time passed, with no change in the disposition of forces. Then, during a feast to Diana, the city inhabitants, "given up entirely to wine and sport," became careless about guarding a section of the wall, and the opportunistic Romans saw their chance. Their armies broke through the lightly guarded section and poured into the city in a vicious and destructive mood. Marcellus, surveying the beautiful town, is said to have wept in anticipation of the havoc that his men were sure to wreak. Indeed, history records that the Romans treated Syracuse no less harshly than they would treat Carthage some 66 years later.

But it was the death of Archimedes that brought Marcellus his greatest sorrow, for he had come to respect his gifted antagonist. According to Plutarch,

> . . . as fate would have it, intent upon working out some problem by a diagram, and having fixed his mind alike and his eyes upon the subject of his speculation, [Archimedes] never noticed the incursion of the Romans, nor that the city was taken. In this transport of study and contemplation, a soldier, unexpectedly coming up to him, commanded him to follow to Marcellus; which he declining to do before he had worked out his problem to a demonstration, the soldier, enraged, drew his sword and ran him through.

Thus ended the life of Archimedes. He died, as he had lived, lost in thought about his beloved mathematics. We can regard him either as a martyr to his research or as a victim of his own preoccupied mind. In any case, mathematicians may come and mathematicians may go, but no other has had an end quite like this.

For all of Archimedes' great weapons, for all of his practical inventions, his true love was pure mathematics. His levers and pulleys and catapults were mere trifles compared with the beautiful theorems he discovered. Again, we quote Plutarch:

> Archimedes possessed so high a spirit, so profound a soul, and such treasures of scientific knowledge, that though these inventions had now obtained him the renown of more than human sagacity, he yet would not deign to leave behind him any commentary or writing on such subjects; but, repudiating as sordid and ignoble the whole trade of engineering, and every sort of art that lends itself to mere use and profit, he placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life.

It was his mathematics that would be his greatest legacy. In this arena, Archimedes stands unchallenged as the greatest mathematician of antiquity. His results, which survive in a dozen books and fragments, are of the highest quality and show a logical sophistication and polish that is truly astounding. Not surprisingly, he was very familiar with Euclid and proved to be a master of Eudoxus' method of exhaustion; to use Newton's charming phrase, Archimedes surely stood on the shoulders of giants. But past influences, great as they were, cannot adequately explain the amazing advances that Archimedes would bring to the discipline of mathematics.

Great Theorem: The Area of the Circle

Around 225 B.C., Archimedes produced a short treatise titled *Measurement of a Circle*, the first proposition of which gave a penetrating analysis of circular area. Before addressing this classic work, however, we first need to examine what was known about circular areas when Archimedes arrived upon the scene.

Geometers of the time would have known that, regardless of the circle in question, the ratio of the circumference of a circle to its diameter is always the same. In modern terminology, we would say that

$$\frac{C_1}{D_1} = \frac{C_2}{D_2}$$

where C is the circumference and D is the diameter of the circles in Figure 4.1. Put another way, the ratio of a circle's circumference to its diameter is constant, and modern mathematicians *define* π to be this ratio. (Note that the Greeks did not use the symbol in this context.) Thus, the formula

$$\frac{C}{D} = \pi \quad \text{or its equivalent} \quad C = \pi D$$

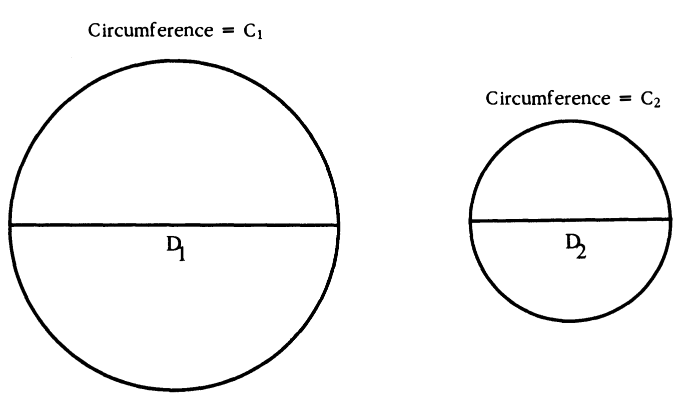

FIGURE 4.1

is nothing more than the definition of the constant π as it arises in the comparison of two lengths—a circle's circumference and its diameter.

But what about circular areas? As we have seen, Proposition XII.2 of the *Elements* established that two circular areas are to each other as the squares on their diameters, and thus the ratio of circular area to the square of the diameter is constant. In modern terms, Euclid had proved that there is some constant k such that

$$\frac{A}{D^2} = k \quad \text{or equivalently} \quad A = kD^2$$

All of this was fine as far as it went. But how do these constants relate to one another? That is, can one find a simple connection between the "one-dimensional" constant π (used in relating circumference to diameter) and the "two-dimensional" constant k (used in relating area to diameter)? Apparently Euclid had found no such connection.

But in his short yet elegant treatise *Measurement of a Circle*, Archimedes proved what amounts to the modern formula for circular area involving π. In doing this, he made the critical link between circumference (and hence π) and circular area. His proof required two fairly direct preliminary results plus a rather sophisticated logical strategy called double *reductio ad absurdum* (reduction to absurdity).

We shall examine these preliminaries first. One concerned the area of a regular polygon with center O, perimeter Q, and apothem h, where the apothem is the length of the line drawn from the polygon's center perpendicular to any of the sides.

THEOREM The area of the regular polygon is $\frac{1}{2}hQ$.

PROOF Suppose the polygon in Figure 4.2 has n sides, each of length b. Draw lines from O to the vertices, thereby breaking it up into a collection of n congruent triangles, each with height h (the apothem) and base b. Since each triangle has area $\frac{1}{2}bh$,

Area (regular polygon)

$$= \tfrac{1}{2}bh + \tfrac{1}{2}bh + \ldots + \tfrac{1}{2}bh, \text{ where the sum contains } n \text{ terms}$$
$$= \tfrac{1}{2}h(b + b + \ldots + b) = \tfrac{1}{2}hQ$$

since $(b + b + \ldots + b)$ is the perimeter.

Q.E.D.

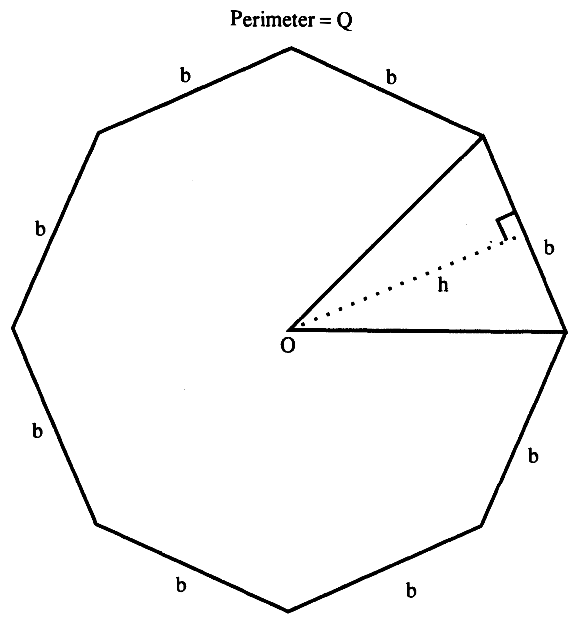

FIGURE 4.2

That was quick enough. Archimedes' other preliminary was also well known in his day, and seems quite self-evident. It says that if we are given a circle, we can inscribe within it a square; Euclid himself gave this construction in Proposition IV.6. The square's area, of course, is less than that of the circle in which it was inscribed. By bisecting each side of the square, we can locate the vertices of a regular octagon inscribed within the circle. Of course, the octagon more nearly approximates the circle's area than the square did. If we again bisected to get a regular 16-gon, it would be closer to the circle in area than the octagon was.

The process can be continued indefinitely. This is, in fact, the essence of Eudoxus' famous method of exhaustion alluded to earlier. Clearly the area of an inscribed polygon never equals that of the circle; there will always be an excess of circle over inscribed polygon regardless of the number of sides of the latter. But—and this was the key to the method of exhaustion—if we have any *preassigned* area, no matter how small, we can construct an inscribed regular polygon for which the difference between the circle's area and the polygon's is less than this preassigned amount. For instance, if we were given a preassigned area of $\frac{1}{500}$ of a square inch, we could come up with a regular inscribed polygon for which

$$\text{Area (circle)} - \text{Area (polygon)} < \tfrac{1}{500} \text{ square inch}$$

That such a polygon might have hundreds or thousands of sides is immaterial; the crucial fact is that it *exists.*

An analogous rule holds for circumscribed polygons. We can summarize both by saying that, for any given circle, we can find polygons—inscribed or circumscribed—whose areas are as close to the circle's area as we want. It is the "close as we want" part of this that held the key to Archimedes' success.

These, then, were his two preliminary propositions. Now a word is needed about the logical ploy he adopted for showing that one area equals another. In some ways this strategy is more sophisticated, or at least more devious, than any we have yet seen. Recall, for instance, how Euclid proved that the square on the hypotenuse equaled the sum of the squares on the legs: he attacked the matter directly, showing that the areas in question were the same. His proof, although extremely clever, was a frontal assault.

But when Archimedes approached the far more complicated circular area, he employed an indirect attack. He realized that, for any two quantities A and B, one and only one of the following cases holds: $A < B$ or $A > B$ or $A = B$. Wanting to prove that $A = B$, Archimedes would first make the assumption that $A < B$ and from this derive a logical contradiction, thereby eliminating the case as a possibility. Next, he would suppose that $A > B$, which again led him to a contradiction. With both of these options eliminated, there remained but one alternative, namely, that A and B are equal.

This was his wonderful, indirect strategy—a "*double reductio ad absurdum*" since it reduced two of the three cases to a contradiction. While this may initially seem a bit roundabout, a little reflection shows it to be quite reasonable; eliminate two of the three possible cases and one is forced to conclude that the third is valid. Certainly no one used double *reductio ad absurdum* more deftly than Archimedes.

With these preliminaries behind us, we can now watch a master at work in the first proposition from *Measurement of a Circle*:

PROPOSITION 1 The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

PROOF Archimedes began with two figures (Figure 4.3): a circle having center O, radius r, and circumference C; and a right triangle having base of length C and height of length r. We denote by A the area of the circle

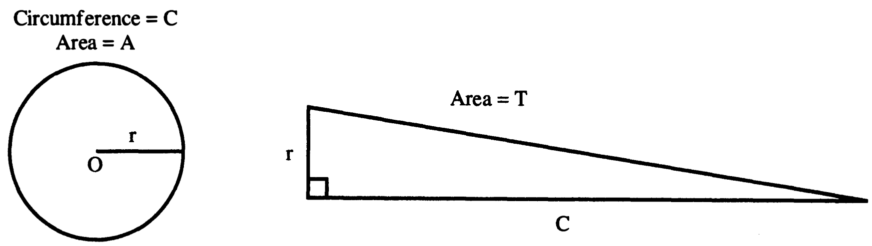

FIGURE 4.3

and by T the area of the triangle. While the former is the object of Archimedes' proof, it is clear that the triangle's area is just $T = \frac{1}{2}rC$.

The proposition claimed simply that $A = T$. To establish this by a double *reductio ad absurdum* proof, Archimedes needed to consider, and eliminate, the other two cases.

CASE 1 Suppose $A > T$.

This asserts that the circular area exceeds that of the triangle by some amount. In other words, the excess $A - T$ is some positive quantity. Archimedes knew that, by *inscribing* a square within his circle and repeatedly bisecting its sides, he could arrive at a regular polygon inscribed within the circle whose area differs from the area of the circle by less than this positive amount $A - T$. That is,

$$A - \text{Area (inscribed polygon)} < A - T$$

Adding the quantity "Area (inscribed polygon) $+ T - A$" to both sides of this inequality yields

$$T < \text{Area (inscribed polygon)}$$

But this is an *inscribed* polygon (Figure 4.4). Thus its perimeter Q is less than the circle's circumference C, and its apothem h is certainly less than the circle's radius r. We conclude that

$$\text{Area (inscribed polygon)} = \tfrac{1}{2}hQ < \tfrac{1}{2}rC = T$$

Here Archimedes had reached the desired contradiction, for he had found both that $T <$ Area (inscribed polygon) and that Area (inscribed polygon) $< T$. There is no logical recourse other than to conclude that Case 1 is impossible; the circle's area cannot be more than the triangle's.

This left him with the second case.

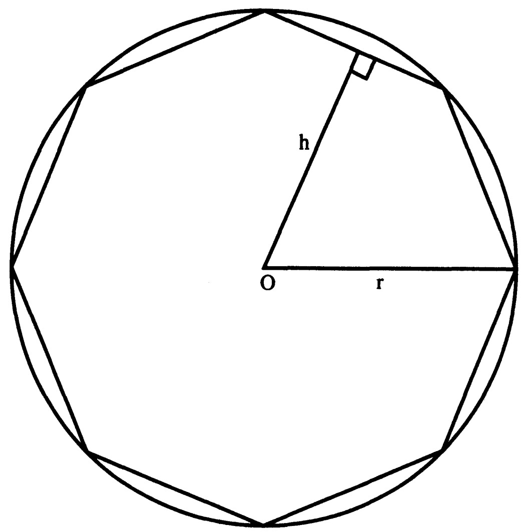

FIGURE 4.4

CASE 2 Suppose $A < T$.

This time Archimedes assumed that the circle's area fell short of the triangle's, so that $T - A$ represented the excess area of the triangle over the circle. We know that we can circumscribe about the circle a regular polygon whose area exceeds the circle's area by less than this amount $T - A$. In other words,

$$\text{Area (circumscribed polygon)} - A < T - A$$

If we simply add A to both sides of the inequality, we conclude that

$$\text{Area (circumscribed polygon)} < T$$

But the circumscribed polygon (Figure 4.5) has its apothem h equal to the circle's radius r, while the polygon's perimeter Q obviously exceeds the circle's circumference C. Thus,

$$\text{Area (circumscribed polygon)} = \tfrac{1}{2}hQ > \tfrac{1}{2}rC = T$$

Again this is a contradiction, since the circumscribed polygon cannot be both less than and greater than the triangle in area. Archimedes con-

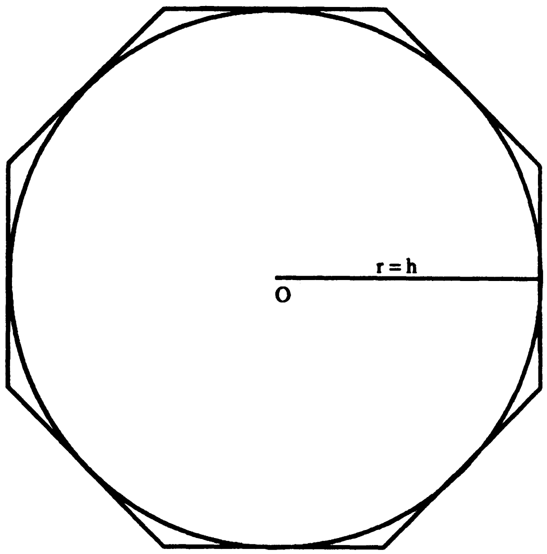

FIGURE 4.5

cluded that Case 2 was likewise impossible; the circle's area cannot be less than the triangle's.

As a consequence, Archimedes could write: "Since then the area of the circle is neither greater nor less than [the area of the triangle], it is equal to it."

Q.E.D.

This was his proof, a little gem from the hand of an indisputably great mathematician. It strikes some people as odd that Archimedes proved the circle's area must equal that of the triangle by showing that it could be neither greater nor less. For those who find his argument a bit too indirect for their taste, a paraphrase of *Hamlet's* Polonius is offered: "though this be madness, yet there is method of exhaustion in't." One is tempted to wonder how something this short and simple could have been overlooked by Hippocrates or Eudoxus or Euclid. But simplicity is most easily perceived in hindsight. In this regard, we again turn to Plutarch's characterization of Archimedes' mathematics:

> It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his natural genius; while others think that incredible effort and toil produced these, to all appearances, easy and unlaboured results. No amount of investigation of yours would succeed in attaining the proof, and yet, once seen, you immediately believe you would have discovered it; by so smooth and so rapid a path he leads you to the conclusion required.

Given that Archimedes had equated the area of the circle with that of a triangle, did he therefore accomplish the long-sought quadrature of the circle that we examined in Chapter 1? The answer of course is "No," for we recall that a successful quadrature requires us to *construct* the rectilinear figure of equal area. Archimedes' proof did not, nor did it claim to, give any inkling as to how to construct the triangle in question. There is, of course, no difficulty in constructing the leg of the triangle equaling the circle's radius; the snag occurs when one tries to construct the other leg equal to the triangle's circumference. Since $C = \pi D$, constructing the circumference amounts to constructing π. As we have seen, no such construction is possible. Archimedes' proof must not be construed as his attempt to square the circle; it was no such thing.

All of this notwithstanding, the reader may yet fail to recognize the familiar formula for the area of a circle in Archimedes' theorem. After all, what he proved was that the area of a circle equaled that of a certain triangle. As we shall see, this was a typical Archimedean device—to relate the area of an unknown figure with that of a simpler, known one. But more was going on than just this. For the triangle in question had as its base the circle's circumference, and this had two crucial implications. First, unlike Euclid, Archimedes had related a circle's area not to that of another circle (basically a "relativistic" approach) but to its own circumference and radius, as reflected in the equivalent triangle. Then, by proving that $A = T = \frac{1}{2}rC$, Archimedes had provided the link between the one-dimensional concept of circumference and the two-dimensional concept of area. Remembering that $C = \pi D = 2\pi r$, we rephrase his theorem as

$$A = \tfrac{1}{2}rC = \tfrac{1}{2}r(2\pi r) = \pi r^2$$

and here emerges one of geometry's most familiar and important formulas.

It is also worth noting that Archimedes' bold proposition easily implied Euclid's relatively tame result that the areas of two circles are in the same ratio as the squares upon their diameters. That is, if we let one circle have area A_1 and diameter D_1 and a second circle have area A_2 and diameter D_2, then Archimedes proved

$$A_1 = \pi r_1^2 = \pi(D_1/2)^2 = \pi D_1^2/4 \quad \text{and} \quad A_2 = \pi r_2^2 = \pi(D_2/2)^2 = \pi D_2^2/4$$

Hence

$$\frac{A_1}{A_2} = \frac{\pi D_1^2/4}{\pi D_2^2/4} = \frac{D_1^2}{D_2^2}$$

which is Euclid's theorem in a nutshell. So, this Archimedean proposition had enough power to imply the Euclidean result as a trivial corollary. Such is the mark of a genuine mathematical advance.

If we look back at the previous discussion, we can now determine the value of the constant k in the "Euclidean" expression $A = kD^2$. For, with Archimedes' discovery at hand, we know that

$$\pi r^2 = A = kD^2 = k(2r)^2 = 4kr^2$$

Hence, $4k = \pi$, and so $k = \pi/4$. In other words, Euclid's "two-dimensional" area constant is just a quarter of π, the "one-dimensional" circumference constant. Thus, his proposition brought the welcome news that we need not calculate two different constants. If we can just determine the value of π from the circumference problem, it would also serve in the formula for circular area.

This latter observation was not lost on Archimedes. In fact, as the third proposition of *Measurement of a Circle*, he derived just such a value.

PROPOSITION 3 The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.

In modern notation, this says: $3\frac{10}{71} < \pi < 3\frac{1}{7}$. With these fractions converted to their decimal equivalents, Archimedes' result becomes 3.140845 . . . $\pi < 3.142857$ . . .; hence, the constant π has been nailed down, to two decimal place accuracy, as 3.14.

That Archimedes came up with this estimate is another sign of his powers. His plan of attack was again to use his ever-helpful inscribed and circumscribed regular polygons, except this time, instead of tracking down their areas, he was concentrating on their perimeters. He began with a regular hexagon inscribed in a circle (Figure 4.6). He knew well that each side of the hexagon equaled the circle's radius, whose length we can call r. Thus,

$$\pi = \frac{\text{circumference of circle}}{\text{diameter of circle}} > \frac{\text{perimeter of hexagon}}{\text{diameter of circle}} = \frac{6r}{2r} = 3$$

Admittedly, this was a very crude estimate for π, but Archimedes had just begun. He next doubled the number of sides of his inscribed polygon, to get a regular dodecagon whose perimeter he had to calculate. This is where he leaves modern mathematicians shaking their heads in wonder, for determining the dodecagon's perimeter required getting a numerical value for the square root of three. With our calculators and

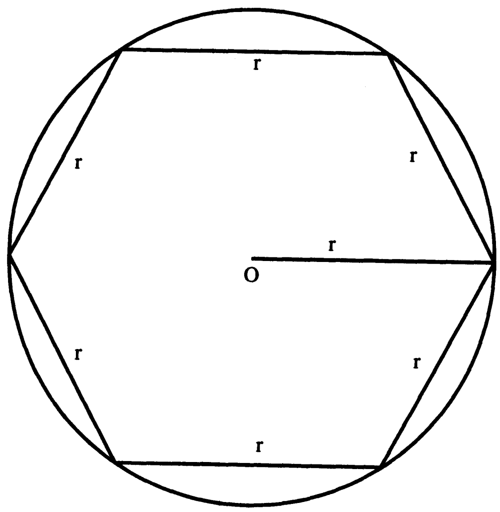

FIGURE 4.6

computers, this strikes us as no real obstacle, but in Archimedes' time, not only were these devices unthinkable, but there was not even a good number system to facilitate such computations. Yet he emerged with the estimate

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

which is impressively close.

From there, Archimedes continued, bisecting again to get a regular 24-gon, then a regular 48-gon, and finally a regular 96-gon. At each stage, he needed to approximate sophisticated square roots, yet he never faltered. When he reached the 96-gon, his estimate was

$$\pi = \frac{\text{circumference of circle}}{\text{diameter of circle}} > \frac{\text{perimeter of regular 96-gon}}{\text{diameter of circle}} > \frac{6336}{2017\frac{1}{4}} > 3\tfrac{10}{71}$$

As if this were not enough, Archimedes then turned around and made similar estimates for regular *circumscribed* 12-gons, 24-gons, 48-gons, and 96-gons, leading him to his upper bound for π of $3\frac{1}{7}$. Such calculations, in the face of an absolutely terrible numeral system and without easy procedures for estimating the square roots he needed, provide sure evidence of his awesome powers. These computations were the arithmetical counterpart of running the high-hurdles wearing a ball and chain. Yet by marshaling his enormous intellect and perseverance, he succeeded in giving the first scientific estimate of the critical constant π. As indicated in the Epilogue to this chapter, the quest for highly accurate estimates of this number has occupied mathematicians ever since.

As it has come down to us, *Measurement of a Circle* contains only three propositions and covers only a few pages of text. Moreover, the second proposition is out of place and unsatisfactory, undoubtedly the result of bad copying, bad editing, or bad translating years, if not centuries, after Archimedes. On the surface, then, it seems unlikely that such a short work would carry the impact that it does. But considering that in its first proposition, Archimedes proved the famous formula for the area of a circle, and in its last, he gave a remarkable estimate for the number π, there is really no doubt why this little treatise had been held in such high regard by generations of mathematicians. It is not the quantity of pages but the quality of the mathematics, and by this criterion *Measurement of a Circle* stands as a genuine classic.

Archimedes' Masterpiece: *On the Sphere and the Cylinder*

The results just discussed constitute but a fraction of the mathematical legacy of Archimedes. He also wrote about the geometry of spirals and about conoids and spheroids, and he provided a remarkable means of finding the area under a parabola by summing a certain infinite geometric series. This latter topic—finding areas under curves—is now treated in calculus courses, another indication (if one were needed) of how utterly far ahead of his time Archimedes was.

But for all of these accomplishments, his undisputed masterpiece was an extensive, two-volume work titled *On the Sphere and the Cylinder*. Here, with almost superhuman cleverness, he determined volumes and surface areas of spheres and related bodies, thereby achieving for three-dimensional solids what *Measurement of a Circle* had done for two-dimensional figures. It was a stunning triumph, one that Archimedes himself seems to have regarded as the apex of his career.

We should first recall what the Greeks knew about the surface areas and volumes of three-dimensional bodies. As noted in the previous chapter, Euclid had proved that the volumes of two spheres are to each other as the cubes of their diameters; in other words, there exists a "volume constant" m so that

$$\text{Volume (sphere)} = mD^3$$

This was the Euclidean treatment of spherical volume. As to the surface area of a sphere, Euclid was utterly silent. Here again, a successful assault on the problem awaited Archimedes' *On the Sphere and the Cylinder.*

This two-volume work had a familiar ring to it, insofar as it began with a list of definitions and assumptions from which he derived ever more sophisticated theorems. In short, it was cast in the Euclidean mold. Its first proposition was the innocuous: "If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the circumference of the circle." However, Archimedes quickly moved in more sophisticated directions. Throughout, he was (at least to modern tastes) hampered by the lack of a concise algebraic notation. Unable to express his volumes and surface areas by simple formulas, he had to rely on statements such as:

PROPOSITION 13 The surface of any right circular cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side of the cylinder and the diameter of the base.

At first glance, this looks quite mysterious and unfamiliar, but it is in the phrasing, not the content, that the unfamiliarity lies. Without the benefit of algebra, Archimedes had to express his desired area—in this case that of a lateral surface of a right circular cylinder—as being equal to the area of a known figure—in this case, a circle (Figure 4.7). But which circle? Obviously Archimedes had to specify his equivalent circle, and that is where the statement about mean proportionals came in.

In modern terminology, Archimedes was claiming that

Lateral surface (cylinder of radius r and height h)
= Area (circle of radius x)

where $h/x = x/2r$. From this it follows quickly that $x^2 = 2rh$, and so we get the well-known formula:

$$\text{Lateral surface (cylinder)} = \text{Area (circle)} = \pi x^2 = 2\pi rh$$

Archimedes proceeded through a string of like-sounding propositions as he approached his first major objective, the surface area of the sphere. Space does not allow us to follow him in his reasoning, but we can acknowledge his remarkable ingenuity. In light of our earlier examination of his mathematics, the reader should not be surprised to learn that Archimedes again used the method of exhaustion. That is, he "exhausted" the sphere by approximating it from within and without by

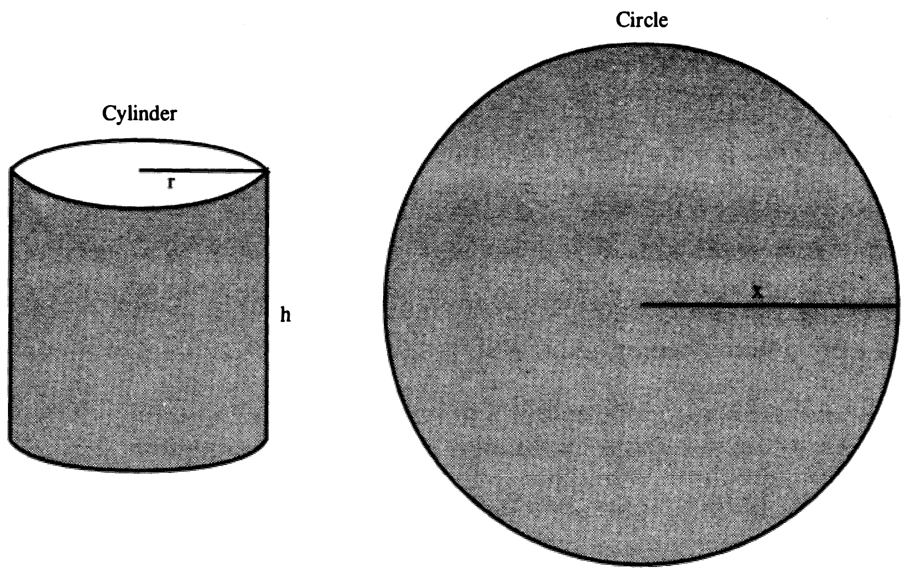

FIGURE 4.7

cones and the frusta of cones, all of whose surface areas he had previously determined. When the dust had settled, he had proved the remarkable

PROPOSITION 33 The surface of any sphere is equal to four times the greatest circle in it.

Archimedes completed the proof with his favorite logical tactic of double *reductio ad absurdum*; that is, he proved it impossible for the spherical surface to be *more* than four times the area of its greatest circle and also proved it impossible for it to be *less* than four times the area of its greatest circle. If we observe that the area of the "greatest circle" of the sphere—that is, the circle through the sphere's "equator"—is just πr^2, then we can translate Archimedes' formulation of this result—"the surface of the sphere is four times the area of its greatest circle"—into the modern-day formula

$$\text{Surface area (sphere)} = 4\pi r^2$$

This is a very sophisticated piece of mathematics. The deftness with which Archimedes handled his concepts, the insights that he brought to bear, seem to anticipate the ideas of modern integral calculus. It is readily apparent why Archimedes is regarded as the greatest mathematician of ancient times.

But there is one other fact about this result that warrants a comment, namely, its utter strangeness. There is nothing intuitive about the substantive fact that the surface of a sphere is *exactly* four times as large as the area of its greatest cross section. Why could it not have been 4.01 times as great? What is so magical about this number "four" to guarantee that if one were to paint the curving surface of a sphere, it would take precisely four times as much paint as it would to paint the great circle through the center?

Archimedes himself addressed this peculiar, intrinsic property of the sphere in his introduction to *On the Sphere and the Cylinder*, which he wrote for a certain "Dositheus," presumably a mathematician at Alexandria to whom Archimedes had sent the treatise. Archimedes noted that ". . . certain theorems not hitherto demonstrated have occurred to me, and I have worked out the proofs of them." First among those he mentioned was ". . . that the surface of any sphere is four times its greatest circle," and he went on to observe that such properties were

> . . . all along naturally inherent in the figures referred to, but remained unknown to those who were before my time engaged in the study of geometry. Having, however, now discovered that the properties are true . . . , I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established . . .

The comment provides an interesting glimpse of Archimedes' assessment of his work and its place in the development of mathematics. He did not hesitate to include himself alongside the great Eudoxus, for he surely was well aware of the extraordinary nature and quality of his own discoveries. But he also went out of his way to stress that he had not invented or created the fact that $S = 4\pi r^2$. Rather, he had been fortunate enough to *discover* an intrinsic property of spheres, one that had existed since time immemorial even though it had been previously unknown to geometers. To Archimedes, mathematical relationships existed independent of the poor efforts of humans to decipher them. He himself had just been the individual fortunate enough to glimpse these eternal truths.

If *On the Sphere and the Cylinder* had contained nothing but the previous theorem, it would have stood as a classic for all time. But he immediately turned his gaze toward spherical *volume*. After another intricate double *reductio ad absurdum* argument, Archimedes succeeded in establishing

PROPOSITION 34 Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.

Note that, again, Archimedes has expressed the volume of the sphere not as a simple algebraic formula but in terms of the volume of a simpler solid, in this case, a cone (Figure 4.8). With just a bit of effort we can convert his verbal statement into its modern equivalent.

That is, let r be the radius of the sphere. Then the "cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere" is such that

$$\text{Volume (cone)} = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi r^2 r = \tfrac{1}{3}\pi r^3$$

But Archimedes' Proposition 34 had proved that the volume of the sphere is four times as great as the volume of one of these cones, and this yields the famous formula

$$\text{Volume (sphere)} = 4\ \text{Volume (cone)} = \tfrac{4}{3}\pi r^3$$

Among its benefits, this result clarifies the link between π and the "volume constant" m that arose from Euclid's Proposition XII.18. Referring to our discussion above, we immediately see that

$$\tfrac{4}{3}\pi r^3 = \text{Volume (sphere)} = mD^3 = m(2r)^3 = 8mr^3$$

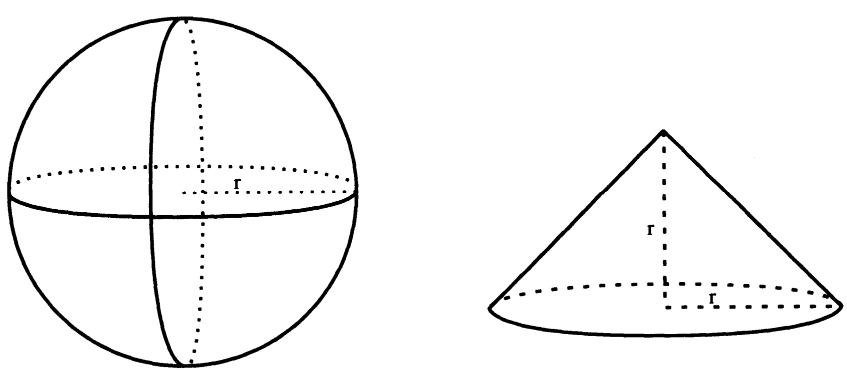

FIGURE 4.8

and a little algebra reveals that $m = \pi/6$. In this fashion, the pre-Archimedean mystery regarding circumferences, circular areas, and spherical volumes was resolved. No longer were three different constants needed to address these three different matters; all three rested upon knowledge of π. Archimedes had exhibited a stunning unity among them.

Immediately upon completing his proofs of Propositions 33 and 34, Archimedes restated his results in a particularly intriguing way. He considered a cylinder circumscribed about the sphere, as shown in Figure 4.9. He then asserted that the cylinder is half again as large as the sphere in *both surface area and volume*! In a certain sense, this was the climax of his whole work. It took his two great results and presented them in a simple fashion, expressing the complicated spherical surface and volume in terms of the correspondingly simpler surface and volume of a related cylinder. This section will conclude with a verification of Archimedes' striking claim.

First, notice that a cylinder circumscribed about a sphere of radius r itself has radius r and height $h = 2r$. The cylinder's overall surface area is the sum of the lateral surface (as in Proposition 13), as well as the circular areas of the top and bottom. Thus,

$$\begin{aligned}\text{total cylindrical surface} &= 2\pi rh + \pi r^2 + \pi r^2 \\ &= 2\pi r(2r) + 2\pi r^2 = 6\pi r^2 \\ &= \tfrac{3}{2}(4\pi r^2) \\ &= \tfrac{3}{2}(\text{spherical surface})\end{aligned}$$

which is precisely what Archimedes meant by saying that the cylinder was "half again" the sphere in surface area.

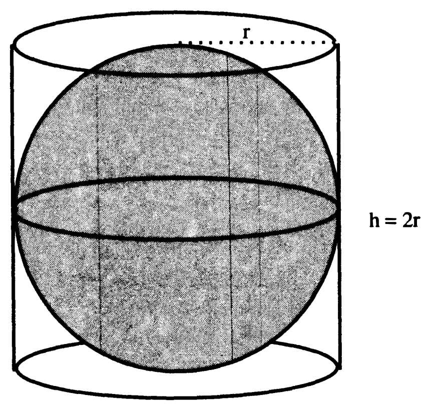

FIGURE 4.9

And what about the corresponding volumes? For a general cylinder, we have $V = \pi r^2 h$, which in this case becomes $V = \pi r^2(2r) = 2\pi r^3$. Thus,

$$\begin{aligned}\text{Cylindrical volume} &= 2\pi r^3 \\ &= \tfrac{3}{2}(\tfrac{4}{3}\pi r^3) = \tfrac{3}{2}(\text{spherical volume})\end{aligned}$$

so that the cylinder was half again the sphere in volume.

Thus, in one concise and remarkable statement, Archimedes had linked the sphere and the cylinder. It was this link that surely accounted for the title of the treatise we are examining. That Archimedes took particular pride in this discovery was indicated by Plutarch's reference to Archimedes' choice of epitaph:

> His discoveries were numerous and admirable; but he is said to have requested his friends and relations that, when he was dead, they would place over his tomb a sphere contained in a cylinder, inscribing it with the ratio which the containing solid bears to the contained [i.e., the ratio 3:2].

Interestingly, Cicero reported in his *Tusculan Disputations* that when in Syracuse he indeed came upon Archimedes' tomb. Admittedly, "a jumble of brambles and bushes" had grown up in the area, concealing everything. But Cicero knew what he was looking for and was understandably excited when he recognized "a small column that emerged a little from the bushes: it was surmounted by a sphere in a cylinder." Having discovered the monument, he took pains to reverse the disrepair into which it had fallen. If true, Cicero had found the final resting place of the greatest of Greek mathematicians. In attempting to rescue the site from oblivion, Cicero not only paid homage to Archimedes but perhaps atoned somewhat for the brutality of his murderous Roman ancestors.

One often hears of people who are ahead of their time. By this is usually meant a man or woman who anticipates the rest of the world by a decade or perhaps even a generation. But Archimedes was doing mathematics whose brilliance would be unmatched for centuries! Not until the development of calculus in the latter years of the seventeenth century did people advance the understanding of volumes and surface areas of solids beyond its Archimedean foundation. It is certain that, regardless of what future glories await the discipline of mathematics, no one will ever again be 2000 years ahead of his or her time.

We can do no better than to end with Voltaire's fitting and quite remarkable comment on the achievements of this great mathematician: "There is more imagination in the head of Archimedes than in that of Homer."

Epilogue

One legacy of Archimedes' *Measurement of a Circle* was the quest for ever more precise estimates of the critical constant we call π. The importance of this ratio had been recognized long before Archimedes, although it was he who first subjected it to a scientific scrutiny. One interesting pre-Archimedean estimate can be inferred from a Biblical quotation about a circular "sea," that is, a large container for holding water: "Then He made the molten sea, ten cubits from brim to brim, while a line of 30 cubits measured it around" (I Kings 7:23).

From this we derive the value $\pi = C/D = 30/10 = 3.00$, an estimate which, because of its great antiquity, is quite reasonable. (Of course, here we have a bone to pick with those who regard the Bible as accurate in *all* respects, since 3.00 seriously underestimates π.)

A better ancient estimate was that of the Egyptians. In the Rhind papyrus, they used $(4/3)^4 = 256/81 = 3.1604938\ldots$ as the ratio of C to D. These and other "pre-scientific" estimates represented the first phase in the estimation of π. As we have seen, Archimedes initiated the second phase. His geometric approach, employing the perimeters of inscribed and/or circumscribed regular polygons, was the method of choice for mathematicians until the mid-seventeenth century (yet another indication that Archimedes was ahead of his time).

Around A.D. 150 the noted astronomer and mathematician Claudius Ptolemy of Alexandria provided an estimate for this ratio in his masterpiece, the *Almagest.* This extensive work was a compilation of astronomical information, from the behavior of the sun and moon, to the motions of the planets, to the nature of the fixed stars in the heavens. Obviously, the precise measurements of celestial objects required a sophisticated mathematical underpinning, and for this reason, early in the *Almagest* Ptolemy developed his Table of Chords.

He began with a circle whose diameter was divided into 120 equal parts. If each part has length p, then we can designate the diameter as $120p$, as shown in Figure 4.10. For any central angle α, Ptolemy wanted to find the length of chord AB subtended by this angle. For instance, the chord of a $60°$ angle is just the length of the radius, which is $60p$.

This was an easy one. Finding the chord of $42\frac{1}{2}°$ is far less simple. But, using some clever reasoning and showing an Archimedean knack for computation, Ptolemy generated precisely such a table for all angles from $\frac{1}{2}°$ up to $180°$ in half-degree increments.

Pertinent to our discussion, however, is the fact that he found the chord of $1°$ to be (in modern decimal notation) $1.0472p$. Thus, the perimeter of a regular 360-gon inscribed in this circle is 360 times as great, namely $376.992p$. Although the idea of using regular polygons is

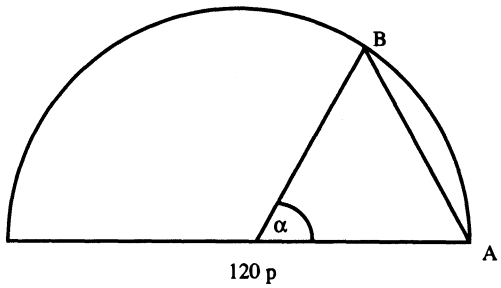

FIGURE 4.10

clearly Archimedean, Ptolemy's 360-sided figure furnished a much more accurate estimate than his predecessor's 96-gon. That is,

$$\pi = \frac{C}{D} \approx \frac{\text{perimeter of 360-gon}}{\text{diameter of circle}} = \frac{376.992p}{120p} = 3.1416$$

In the centuries that followed, advances in the calculation of π centered in the non-Western cultures of China and India, cultures with brilliant mathematical histories of their own. Thus we find the Chinese scientist Tsu Ch'ung-chih (430–501) using the estimate $355/113 = 3.14159292\ldots$ around A.D. 480, and the Hindu mathematician Bhāskara (1114–ca. 1185) recommending $3927/1250 = 3.1416$ for accurate calculations around A.D. 1150.

When Europe finally emerged from the mathematical stagnation of the Middle Ages, the pace of discovery accelerated. By the late sixteenth century, with the work of such mathematicians as Simon Stevin (1548–1620), the modern decimal system had been established, and with it came easier, more accurate estimates of square roots. Thus, when the gifted French mathematician Francois Viète (1540–1603) tried his hand at estimating π with Archimedes' technique, he could use regular polygons of 393,216 sides to get a value accurate to nine places. This required him to follow Archimedes' lead through the 96-gon, but then to double the number of sides a dozen more times. Even Archimedes would have withered under the constraints of his number system, but the decimal notation gave Viète the opening he needed. The basic insight was still Archimedes' but Viète had better tools.

Early in the seventeenth century, a Dutch mathematician outdid all predecessors by finding π correct to 35 places. His name was Ludolph van Ceulen, and he devoted years of effort to the task. Like Viète, Ludolph combined the new decimal system with the old Archimedean strategy, although rather than starting with a hexagon and doubling its number of sides, Ludolph began with a square. By the time he was finished, he was handling regular polygons with 2^{62}—or roughly 4,610,000,000,000,000,000—sides! Needless to say, the perimeter of such a polygon differs very little from the circumference of the circle in which it is inscribed.

The classical method of approximating π had carried mathematicians far. But later in the seventeenth century came a mathematical explosion of epic proportions, one of whose advances at last supplanted Archimedes' approach and pushed the search for π into its third phase. In the late 1660s, the young Isaac Newton applied his generalized binomial theorem and newly invented method of fluxions—that is, calculus—to get a very accurate estimate of π with relative ease; this is the great theorem dealt with in Chapter 7. By 1674, Newton's rival Gottfried Wilhelm Leibniz had discovered that the series

$$1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + 1/13 - 1/15 + \dots$$

approaches the number $\pi/4$ as we carry the calculations ever farther along. Theoretically at least, we can extend the series of terms as far as we choose in order to get ever more accurate approximations to $\pi/4$, and consequently to π itself. It is important to note that the series we must sum here is utterly predictable in its behavior; that is, no matter where we are in the series, it is easy to determine the next term. Suddenly, then, the matter of approximating π turned from the *geometric* problem it had been with Archimedes' regular polygons to a simple *arithmetic* problem of adding and subtracting numerical terms. This was a major change in perspective.

Actually, the plot thickened at this point, since Leibniz's series, while it did indeed approach the number $\pi/4$, did so very slowly. For instance, even if we use the first 150 terms of the series, we get as an approximation of π only 3.1349 . . . , which is disappointingly inaccurate given the number of computations involved. It is estimated that to get 100-place accuracy with this series, one would need more than

100,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000

terms! So, while Leibniz's series foretold the new, arithmetic approach to estimating π, it obviously had little practical use.

The promise of infinite series was soon fulfilled as mathematicians such as Abraham Sharp (1651–1742) and John Machin (1680–1751) made clever modifications that generated much more rapidly converging series. Using these adjustments, Sharp found π correct to 71 places in 1699, and Machin got 100 places seven years later. Moreover, their efforts proved far easier than those which had occupied poor Ludolph for much of his life in squeezing out 35-place accuracy. It was clear that the series approach had rendered the classical method obsolete.

Meanwhile there were developments on other fronts in mathematicians' attempts to understand this peculiar constant. Chief among these was the 1767 proof by Johann Heinrich Lambert (1728–1777) that π is an irrational number. We recall that the irrationals are those real numbers that cannot be written as the quotient of two integers—that is, the irrationals are the numbers that are not fractions. It is fairly easy to show that constants like $\sqrt{2}$ or $\sqrt{3}$ are irrational, but it took until the eighteenth century for Lambert to prove that π belonged on this list. His discovery assumes particular importance when we recall that rational numbers have decimal expansions that either terminate or exhibit a repeating pattern. For instance, the decimal for the rational number $\frac{1}{8}$ is just .125. Alternately, the decimal for the rational $\frac{1}{7}$ never stops, but at least it repeats in blocks of six places:

$$\frac{1}{7} = .142857142857142857142857\ldots$$

If π were rational, it too would have to exhibit one of these behaviors, and thus efforts to determine its decimal expansion would, after a certain amount of time, essentially be complete. Lambert's proof that π belonged among the irrational numbers guaranteed that the computation of its decimal would forever remain unfinished business.

As if this irrationality were not already bad enough, Ferdinand Lindemann proved in 1882 that π is actually transcendental, as mentioned in Chapter 1. Not only did this discovery settle the issue of squaring the circle, but it meant that π could not emerge as any sort of elementary expression involving square roots, cube roots, and so on, of rational numbers. The results of Lambert and Lindemann showed that π is not among the "nice" numbers easily accessible to mathematical analysis. Yet the results of Archimedes from 225 B.C. had shown just as clearly that π was one of the most important numbers of all.

This history of π introduces one of the outstanding mathematicians of this century, Srinivasa Ramanujan (1887–1920). Born in India to a family of limited means, Ramanujan enjoyed none of the benefits of formal mathematical training. He was largely self-taught, and this from just a few textbooks. Ramanujan's absorption with mathematics cost him dearly in his mastery of other subjects, and his formal education ended when he was unable to pass the requisite examinations in neglected courses. By 1912, he was reduced to a clerical job in Madras, supporting himself and his wife on a mere 30 pounds per year. It would have been very easy to write him off as a failure.

Yet, despite such obstacles, this isolated genius was doing mathematical research of great originality and depth. After some urging, he wrote up a sampler of his discoveries and mailed them to three of England's foremost mathematicians. Two of them returned Ramanujan's unsolicited letter. Apparently they felt they had more pressing things to do than to respond to an unknown Indian clerk.

The third, G. H. Hardy of Cambridge University, may have been tempted to follow the same course when he opened his morning mail on January 16, 1913. Ramanujan's communication, written in poor English and containing over 100 strange formulas without proofs of any kind, seemed to be the disordered ramblings of a crackpot from halfway around the world. Hardy put the letter aside.

But, as the story goes, something about those mathematical formulas haunted him all that day. Many of the results were unlike anything Hardy had ever seen, and Hardy was among the finest mathematicians in the world. Gradually, it dawned upon him that these formulas "... must be true, because if they were not true, no one would have had the imagination to invent them." Indeed, when he returned to his rooms and reexamined the morning's document, Hardy realized that this was the work of an enormous mathematical talent.

Thus began the process of bringing Ramanujan to England. It was complicated by a staunch religious upbringing that placed restrictions on his mode of travel, his diet, and so on. But these problems were eventually overcome, and Ramanujan arrived at Cambridge in 1914.

There followed an extraordinary half-decade of collaboration between Ramanujan and Hardy—the latter being a sophisticated, urbane Englishman possessing the best mathematical training the world could offer; the former being a "raw talent" of incredible power who nonetheless had huge gaps in his mathematical knowledge. Sometimes Hardy had to instruct his young companion even as he would an ordinary undergraduate. At other times Ramanujan would astound him with never-before-seen mathematical results.

Among the formulas that Ramanujan devised were many that gave rapid, highly accurate approximations to π. Some of these appeared in an important 1914 paper; others were scrawled in his private notebooks (documents only now being made generally available to the world's eager mathematical community). Even the simplest of these formulas would carry us too far afield, but suffice it to say that his insights have opened lines of investigation into far more efficient estimates of π.

Unfortunately, Ramanujan's career, so improbable in its beginnings, came to a premature end. Far from home, in Cambridge during World War I, Ramanujan suffered a physical breakdown. Some attributed his decline to disease; others saw the cause as a serious vitamin deficiency brought on by his severe dietary restrictions. In the hope of recovery, he returned to India in 1919, but the familiarity of home was unable to arrest his decline. On April 26, 1920, Ramanujan died, and the world lost, at age 32, one of its mathematical legends.

We now rapidly bring our story to its modern conclusion by citing the amazing calculations of the Englishman William Shanks (1812–1882), who determined π to 707 places in 1873. Shanks had used the series approach of Machin to get this startling level of accuracy, which stood as a standard for the next 74 years. But then, in 1946, his countryman D. F. Ferguson made the startling discovery that Shanks had erred after the 527th place of his great computation. Ferguson then kindly corrected the mistake and obtained π to 710 places. For those with less of an appetite for calculations, it is difficult to imagine undertaking a *check* upon a 707-place number; more incredible is the persistence that would keep one going after finding no errors through 100 places, then 200 places, then 500 places! Yet Ferguson's inexplicable perseverance did in fact pay off.

In early 1947, the American J. W. Wrench added his own achievement to this history by publishing π to 808 places. This seemed to be a brilliant new triumph—until the indefatigable Ferguson began checking this one too. Sure enough, he found a mistake in the 723rd place of Wrench's computation. The two men then joined forces and a year later provided π correct to 808 places.

At this point, the tale enters its fourth and final phase. We have seen how people first estimated π by a sort of "rule of thumb"; next, Archimedes introduced the method of inscribed and circumscribed polygons, which prevailed until the coming of calculus when arithmetical techniques involving infinite series took over. Finally, in 1949 the computer fundamentally revolutionized the calculations. In that year, the Army's ENIAC computer found π to 2037 places. It should be stressed that this was, by modern standards, an extremely primitive machine, one which filled rooms with wires and vacuum tubes and cranked out its results with excruciating slowness. Yet even this quaint old device managed to obliterate all previous human calculations, in one leap extending the decimal estimate by two-and-a-half times beyond 22 centuries of human achievement. Not even D. F. Ferguson was going to find an error in this one. Further, as computer technology improved, the number of decimal places grew at an unbelievable pace. By 1959, there were over 16,000 places; by 1966, it had risen to a quarter of a million places, and by the late 1980s, supercomputers had pushed the expansion to somewhere over half a billion places, give or take a few million.

Yet our fragile human egos need not be too severely damaged. For, while the computers are faster at calculations than any person can hope to be, it was mathematicians who programmed the machine and thereby pointed it in the proper direction. The story of π is the story of a human, not a mechanical, triumph. And even in the late twentieth century, we must not forget that this journey had its mathematical beginnings in the short treatise *Measurement of a Circle* by the unsurpassed Archimedes of Syracuse.